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BENDING OF A CIRCULAR PLATE ON A LINEARLY- DEFORMABLE FOUNDATION UNDER A SIMULTANEOUS ACTION OF THE LONGITUDINAL AND TRANSVERSE FORCES

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The problem of bending of a circular plate on an elastic half-space under the simultaneous action of the longitudinal and transverse forces was investigated earlier in [1].

The present paper gives a method of solving the problem of bending of a circular plate on a linearly-deformable foundation. The method makes possible the study of various models of the foundation and of the forms of loading for a wide range of variation of the dimensionless parameters characterizing the flexibility of the plate. The algorithm given in the paper is used to write an "Algol-60" program. An example deals with the bending of a circular plate on an elastic layer resting on a rocky support without friction, and on a layer rigidly joined to a deformable support.

Let a circular plate of radius R lie without friction on a linearly-deformable, general type foundation. The plate is acted upon by a normal load $p^*(r')$ axially distributed over the upper surface, longitudinal forces T and reaction $q^*(r')$ acting from the direction of the support.

In determining the deflections $w^*(r')$ of the plate we shall allow for the possibility of using the Kirchoff-Love theory. The problem can be stated in the form of a system of equations which in dimensionless coordinates has the form ($r' = Rr$, $\rho' = R\rho$)

$$\Delta^2 w(r) - d\Delta w(r) = p(r) - q(r) \tag{1.1}$$

$$\int_0^1 q(\rho)\left[\frac{2\lambda}{\pi(r+\rho)} K\left(\frac{2\sqrt{r\rho}}{r+\rho}\right) + F\left(\frac{r}{\lambda}, \frac{\rho}{\lambda}\right)\right]\rho d\rho = \lambda\mu v(r, 0) \tag{1.2}$$

$$w(r) = v(r, 0), \quad r \leqslant 1$$

$$\lambda = HR^{-1}, \quad \mu = \theta R^3 D^{-1}, \quad w^*(r') = w(r) R, \quad d = TD^{-1}R^2$$

$$v^*(r', 0) = v(r, 0) R, \quad p^*(r') = p(r) DR^{-3}, \quad q^*(r') = q(r) DR^{-3},$$

$$\Delta = \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)$$

Here $v^*(r', 0)$ denotes the settlement of points on the surface of the foundation, D is cylindrical rigidity of the plate, H is a geometrical parameter of the foundation, θ characterizes the physical and mechanical properties of the foundation and $K(e)$ is a complete elliptic integral of the first kind.

The function $F(t, \tau)$ has the form [2,3]

$$F(t, \tau) = \int_0^\infty [L(u) - 1] J_0(t, u) J_0(\tau, u)\, du \tag{1.3}$$

Here $L(u)$ is a function which defines the elastic properties of the foundation, and $J_0(t, u)$ is the Bessel function of zero order. For the basic models of the founda - tion the behavior of the function $L(u)$ at infinity and at zero is governed by the foll - owing relations:

$$L(u) = 1 + O(u^{-2}), \quad u \to \infty; \quad L(u) = O(u^{\gamma}), \quad u \to 0, \quad \gamma \geqslant 1$$

The functions $w(r)$ must satisfy the free edge conditions at the plate contour

$$\Delta w(r) - \frac{1-\nu}{r} \frac{dw(r)}{\partial r}\Big|_{r=1} = 0 \tag{1.4}$$

$$\frac{d}{dr}\left[\Delta w(r) - d\frac{dw(r)}{dr}\right]\Big|_{r=1} = 0$$

(here ν is the Poisson's ratio of the plate material)
The conditions (1.1), (1.2) and (1.4) must be supplemented by the obvious condition of statics

$$\int_0^1 [p(r) - q(r)]\, rdr = 0 \tag{1.5}$$

The boundary value problem just formulated is self-conjugate. This enables us to write the deflection functions in the form [4]

$$w(r) = \sum_{m=0}^{\infty} b_m Q_m(r) \tag{1.6}$$

Here $Q_m(r)$ is a special system of orthonormal polynomials satisfying the boundary conditions (1.4). The condition of their orthogonalizationand normalization [5] is given by

$$\int_0^1 [\Delta^2 Q_m(r) - d\Delta Q_m(r)]\, Q_k(r)\, rdr = \begin{cases} 0, k \neq m \\ 1, k = m, k \geqslant 1 \end{cases} \tag{1.7}$$

We have

$$Q_0(r) = 1, \quad Q_k(r) = \sum_{s=0}^{k+1} l_s(d)\, r^{2s+2} \tag{1.8}$$

Taking into account the linear character of the problem, we seek the solution of the integral equation in the same form as that of the deflection function

$$q(r) = \sum_{m=0}^{\infty} b_m q_m(r) \tag{1.9}$$

The coefficients $b_m(d, \lambda, \mu)$ are round from the equations (1.1) and (1.5), using the condition (1. 7)

$$\gamma_m b_m = \int_0^1 [p(r) - q(r)]\, Q_m(r)\, rdr \tag{1.10}$$

$$\gamma_m = 0 \quad \text{for} \quad m = 0, \quad \gamma_m = 1 \quad \text{for} \quad m \geqslant 1$$

Substituting the expressions (1.6) and (1. 9) into the integral equations (1.2), we obtain the following integral equation for $q_m'(r)$:

$$\int_0^1 q_m(\rho)\left[\frac{2\lambda}{\pi(r+\rho)} K\left(\frac{2\sqrt{r\rho}}{r+\rho}\right) + F\left(\frac{r}{\lambda}, \frac{\rho}{\lambda}\right)\right]\rho d\rho = \lambda\mu Q_m(r) \tag{1.11}$$
$$r \leqslant 1, \quad m = 0, 1, 2, \ldots$$

2. To solve the itegral equation (1. 11.), we reduce it to an infinite linear algebriac system [2, 3]. We write the function $F(t, \tau)$ of (1. 3) in the form of a dual series in even Legendre polynomials, and expand the functions $q_m(\rho)$ and $Q_m(r)$ also into series in Legendre polynomials. Thus we have

$$F\left(\frac{r}{\lambda}, \frac{\rho}{\lambda}\right) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} e_{kj}(\lambda) P_{2k}(\sqrt{1-\rho^2}) P_{2j}(\sqrt{1-r^2}) \tag{2.1}$$

$$q_m(\rho) = \mu \sum_{k=0}^{\infty} S_k^m \frac{P_{2k}(\sqrt{1-\rho^2})}{\sqrt{1-\rho^2}}$$

$$Q_m(r) = \sum_{k=0}^{\infty} R_k^m P_{2k}(\sqrt{1-r^2}), \quad R_k^m = 0 \quad \text{при} \quad k > m+2$$

Using the property of orthogonality of the Legendre polynomials and the integral [6]

$$\int_0^1 J_0(bx) P_{2k}(\sqrt{1-x^2}) \frac{x dx}{\sqrt{1-x^2}} = \sqrt{\frac{\pi}{2}} \frac{(2k-1)!!}{(2k)!!} \frac{1}{\sqrt{b}} J_{2k+1/2}(b)$$

we obtain the following expression for the coefficients $e_{kj}(\lambda)$:

$$e_{kj} = \pi\lambda(4k+1)(4j+1)\frac{(2k-1)!!\,(2j-1)!!}{2\,(2k)!!\,(2j)!!}$$
$$\times \int_0^{\infty} [1 - L(u)] J_{2k+1/2}\left(\frac{u}{\lambda}\right) J_{2j+1/2}\left(\frac{u}{\lambda}\right)\frac{du}{u}$$

For the coefficients R_k^m we have

$$R_k^m = (4k+1)\int_0^1 Q_m(\rho) P_{2k}(\sqrt{1-\rho^2}) \frac{\rho d\rho}{\sqrt{1-\rho^2}}, \quad k \leqslant m+2$$

Substituting the functions $F(t, \tau)$, $q_m(\rho)$ and $Q_m(r)$ of the form (2. 1) into the integral equation (1. 11) and using the spectral relation [3],

$$\int_0^1 \frac{\rho P_{2m}(\sqrt{1-\rho^2})}{\sqrt{1-\rho^2}} K\left(\frac{2\sqrt{r\rho}}{r+\rho}\right)\frac{d\rho}{r+\rho} = \frac{\pi^2}{4}\frac{[(2m-1)!!]^2}{[(2m)!!]^2} P_{2m}(\sqrt{1-r^2})$$

we obtain an infinite system of linear algebraic equations for the coefficients S_k^m

$$S_k^m \frac{\pi}{2}\frac{[(2k-1)!!]^2}{[(2k)!!]^2} = R_k^m + \frac{1}{\lambda}\sum_{n=0}^{\infty} S_n^m \frac{e_{kn}(\lambda)}{4n+1} \tag{2.2}$$

The system (2. 2) was proved in [3] to be quasi-completely regular for all $0 < \lambda < \infty$, and it can be solved using the method of reduction.

Having found the coefficients $S_k^m(\lambda, d)$, we complete the solution of the problem by substituting the second expression of (2. 1) into (1. 10) and solving the infinite system of linear algebraic equations

$$\gamma_m b_m + \sum_{s=0}^{\infty} b_s c_{sm} = f_m, \quad m = 0, 1, 2, \ldots \tag{2.3}$$

$$\gamma_m = 0 \quad \text{for } m = 0, \quad \gamma_m = 1 \quad \text{for } m \geqslant 1$$

$$c_{sm} = \int_0^1 q_s(\rho) Q_m(\rho) \rho d\rho, \quad f_m = \int_0^1 p(r) Q_m(r) r dr$$

It can be shown that the latter system is also quasi-completely regular.

After determining the coefficients b_m the basic parameters of the problem in question are determined using the formulas (1. 6) and (1. 9). Thus the problem reduces to that of solving consecutively the systems (2. 2) and (2. 3) of linear algebraic equations.

3. Using the algorithm given above for solving the problem of bending of circular plates on a linearly deformable foundation of general type under the simultaneous action of the longitudinal and transverse forces, we construct a general numerical Algol-60 program. The relative independence of the separate blocks of the program makes it possible to change at will the model of the foundation, the form of the load $p(r)$, the magnitude of the longitudinal forces T, the relative flexibility μ of the plate, the dimensionless parameter λ and the accuracy of the final results.

As an illustration, we consider two models of the elastic foundation. 1) An elastic layer of finite thickness H lying without friction on a rigid support, and 2) an elastic layer rigidly bound to an undeformable support. The function $L(u)$ for the above problems has the form

$$1) \qquad L(u) = \frac{\operatorname{ch} 2u - 1}{\operatorname{sh} 2u + 2u}$$

$$2) \qquad L(u) = \frac{2\varkappa \operatorname{sh} 2u - 4u}{2\varkappa \operatorname{ch} 2u + 1 + \varkappa^2 + 4u^2}, \quad \varkappa = 3 - 4\nu_1$$

Here ν_1 is the Poisson's ratio of the layer material.

Computations were performed for various types of load. namely a uniform load, a load concentrated at the center, along the edge and concentrated moments along the egde. It was found that when the value of the parameter d was fixed, the convergence of the method improved with decreasing λ. This follows from the fact that when $\lambda \to 0$ the first series of (2. 1) becomes divergent along the line $t = \tau$ [3]. The computations, however, indicate that when $\lambda \geqslant 0.5$ and $\mu < 30$, solutions with three significant figures can be secured provided that 4–8 equations are taken in the system (2. 2) and 3–6 equations in (2. 3). In addition, the smaller the value of λ and the larger the value of μ, the greater the number of equations necessary to achieve the required accuracy. Increasing the parameter d from zero to 15 the convergence of the method improves. Further increase in the value of d or assigning to it negative values have a worsening effect. When the values are negative, one must also consider the problem of stability of the plate.

Under the concentrated loads the same effect is observed, but the order of the systems (2.2) and (2.3) is increased by one.

Computations show that when $\lambda > 4$ while μ and d are arbitrary, the method of attachment of the layer to its support ceases to have any appreciable influence on the results of the computations of the contact pressure and deflection. When $\lambda > 6$ the problem of bending of a plate on an elastic half-space can be studied as a particular case, and the error does not exceed 3%. The computations show good agreement with the results of [1].

Figures 1 and 2 depict the graphs of the contact pressure $q(r)$ for the models 1) and 2) of foundation under a uniform load $p(r) = 1$, $\lambda = 1$, $\nu = 1/_6$, $\nu_1 = 0.3$, $0 = G_1 / (1 - \nu_1)$ (G_1 is the shear modulus of the layer material) and various values of μ and d. Solid lines correspond to $d = 1$, and the dashed lines to $d = 10$.

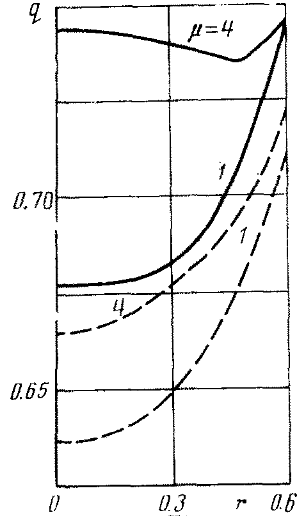

Fig. 1

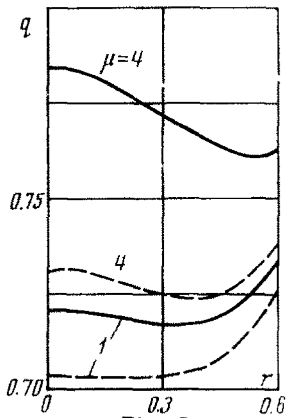

Fig. 2

The computation results show that the plate becomes more rigid with increasing d, and the effect of the tensile (compressive) forces more pronounced with the increasing flexibility of the plate. The tendency is greater when the layer lies freely on a rigid foundation.

When $|d| < 0.5$, the results of the computations for $q(r)$ and $w(r)$ practically coincide with the results obtained for the problem of bending of a plate under the action of a vertical load [7 - 9], i.e. when $|d| < 0.5$, the tensile (compressive) forces can be neglected.

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